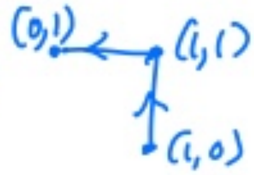
From last time.

① Find $\int_C 2x\cos(y)\,dx - x^2\sin(y)\,dy$

if C is a curve from $(1,0)$ to $(0,1)$

where C is
ⓐ along unit circle
ⓑ along a straight line
ⓒ Along this path



② Same for $\int_C x^2\,dy$

Last time ⓐ Ans $= -1$.

ⓑ Ans $= -1$.

<u>Question</u>: is $V(x,y) = \left(2x\cos(y), -x^2\sin(y)\right)$ conservative?

$$\nabla F = (F_x, F_y)$$

$$\boxed{F(x,y) = x^2\cos(y)}$$

<u>Ans</u>: ⓐ, ⓑ, ⓒ : $F(\text{end pt}) - F(\text{beg pt.})$

$$= F(0,1) - F(1,0)$$

$$= 0^2\cos(y) - 1^2\cos(0) = \boxed{-1}.$$

$F_x = 2x\cos(y) \xRightarrow{\text{antider wrt. } x} F(x,y) = x^2\cos(y) + C(y)$

$F_y = -x^2\sin(y) \xRightarrow[\text{antider wrt } y]{} F(x,y) = x^2\cos(y) + K(x)$

$\Rightarrow F(x,y) = x^2\cos(y) + C(y) = x^2\cos(y) + K(x)$

Can take $C(y) = K(x) = 0$. (or $= 5$)

Homework Example

$$V = (x^2 - y, -x + 2y)$$

Is this conservative?

$V = \nabla F$

$F_x = x^2 - y \Rightarrow F(x,y) = \frac{1}{3}x^3 - yx + C(y)$

$F_y = -x + 2y \Rightarrow F(x,y) = -xy + y^2 + K(x)$

$F(x,y) = \frac{1}{3}x^3 \underline{-yx} + C(y) = \underline{-xy} + y^2 + K(x)$

$\Rightarrow C(y) - y^2 = K(x) - \frac{1}{3}x^3 \overset{\text{set}}{=} 0$

$C(y) = y^2, \quad K(x) = \frac{1}{3}x^3$

$\Rightarrow F(x,y) = \frac{1}{3}x^3 - yx + y^2 = -xy + y^2 + \frac{1}{3}x^3$ ✓

$\therefore V$ is conservative & $V = \nabla F$

$\therefore \oint_\alpha V \cdot ds = F(\text{end pt}) - F(\text{beg pt})$

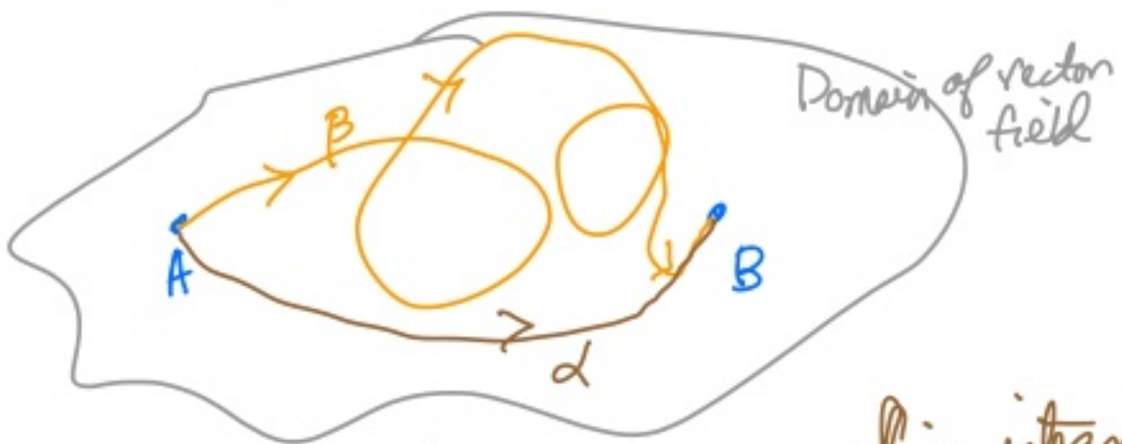
$= F(-1,0) - F(1,0)$

$= \left(\frac{1}{3}(-1)^3 - 0(-1) + 0^2\right) - \left(\frac{1}{3}(1)^3 - 0 + 0\right)$

$$= -\frac{1}{3} - \frac{1}{3} = \boxed{\frac{-2}{3}}.$$

"Line integrals of conservative vector fields are independent of path."

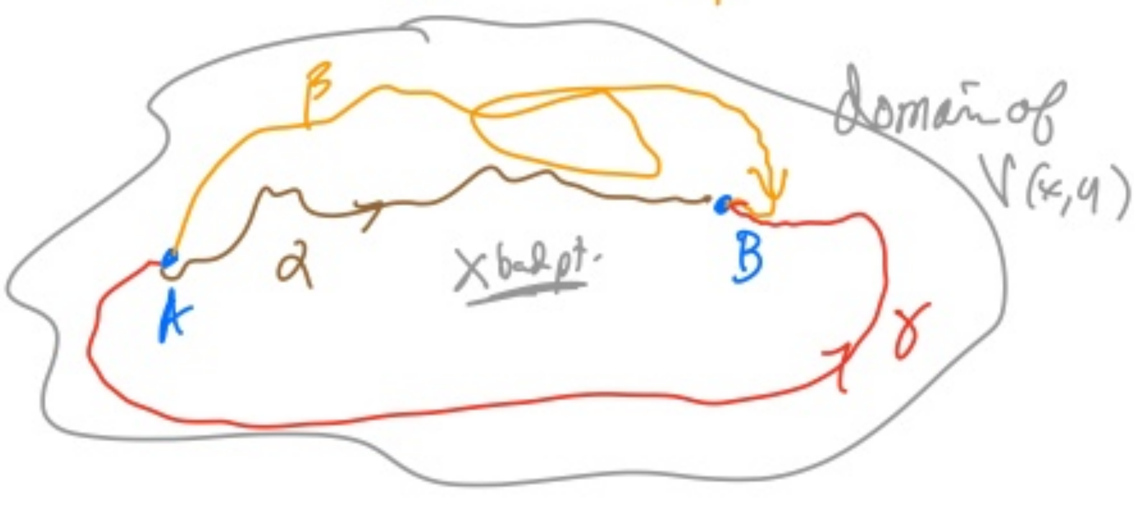
True as long as the two path can continuously deformed through the domain of the vector field.



both give same line integral

$$\oint_\alpha V\cdot ds = \oint_\beta V\cdot d = F(B) - F(A).$$

But:



It could be that $\nabla F_1 = V$ around the upper side

$$\oint_\alpha V\cdot ds = \oint_\beta V\cdot ds = F_1(B) - F_1(A)$$

& $\nabla F_2 = V$ around lower side

$$\oint_\gamma V\cdot ds = F_2(B) - F_2(A)$$

might not be the same.

If the domain of the conservative vector field is simply connected and connected,

No holes (in $\mathbb{R}^2$)

only one piece to the domain.

then every line integral is independent of any path.

Corollary of this: If α is a closed loop, V is a conservative vector field, and the inside of α is inside the domain, then

$$\oint_\alpha V\cdot ds = 0$$

Tests for Conservative Vector Fields:

① $\mathbb{R}^2$ $\quad V = (V_1, V_2)$

want F_x $\quad F_y$

V conservative $\Rightarrow (V_1)_y = (V_2)_x = F_{xy}$

(in simply connected domain)

Examples:

$$V = (2x\cos(y), -x^2\sin(y))$$

$$(2x\cos(y))_y = -2x\sin(y)$$

$$(-x^2\sin(y))_x = -2x\sin(y)$$

$= \checkmark$ conserv.

$$V = (x^2 - y, -x + 2y)$$

$$(x^2 - y)_y = -1$$

$$(-x+2y)_x = -1$$

same $\Rightarrow$ conservative

$\underline{\mathbb{R}^3}$: V is conservative

$$\iff \operatorname{curl} V = 0 = (0,0,0)$$

$$\nabla \times V = 0 = (0,0,0)$$

measures circulation

$$\nabla \times V = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= i\left(\partial_y V_3 - \partial_z V_2\right) - j\left(\partial_x V_3 - \partial_z V_1\right)$$

$$+ k\left(\partial_x V_2 - \partial_y V_1\right)$$

$$= \left(\partial_y V_3 - \partial_z V_2, -\partial_x V_3 + \partial_z V_1, \partial_x V_2 - \partial_y V_1\right)$$

V $\underline{\text{is conservative}}$ $\iff \nabla \times V = (0,0,0).$